Lecture 09
13.3-13.5: Velocity and normal vectors of a curve

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Things to note

Exam 1 is on Monday. Study guide will be posted soon (or has been posted?)

Quiz 04 is today (vector-valued derivatives, integrals, arc length).

Friday will be a review day with no quiz.

Office hours canceled today.

Last class

Definition

Let $\vec{\mathbf{r}}(t) = f(t)\vec{\mathbf{i}} + g(t)\vec{\mathbf{j}} + h(t)\vec{\mathbf{k}}$ *be smooth and let* $a \leq t \leq b$*.*
Then the length of $\vec{\mathbf{r}}$ *from* $t = a$ *to* $t = b$ *is*

$$L = \int_a^b \sqrt{\left(\frac{df}{dt}\right)^2 + \left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2} dt.$$

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The arclength parameter with base point t_0 is

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Example

Find the arclength parameter of the helix with base point 0.

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Example

Find the arclength parameter of the helix with base point 0.

The arclength parameter is

$$s(t) = \int_0^t \sqrt{(-\sin(\tau))^2 + (\cos(\tau))^2 + 1^2} d\tau = \int_0^t \sqrt{1+1} d\tau$$

$$= \sqrt{2}\tau \Big]_0^t = \sqrt{2}t.$$

Unit tangent vector

The are several important unit vectors associated with a smooth space curve. The first is the unit tangent vector.

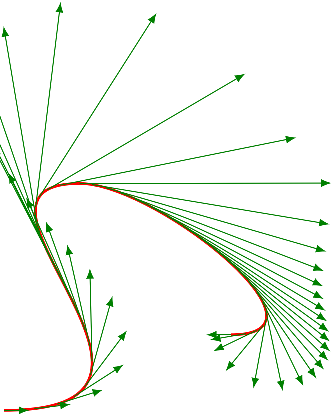
Definition

Let $\vec{\mathbf{r}}(t)$ *be a smooth curve. Let* $\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}}{dt}$*. Then the unit tangent vector* $\overrightarrow{\mathbf{T}}$ *of* $\vec{\mathbf{r}}$ *is*

$$\overrightarrow{\mathbf{T}} := \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|}.$$

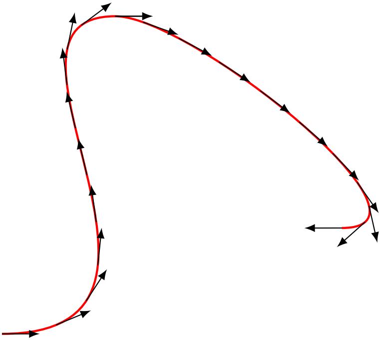
Recall

We can visualize $\frac{d\vec{\mathbf{r}}}{dt}$ geometrically as the tangent vector to the space curve.



Picture

The vector $\vec{\mathbf{T}}$ points in the same direction as $d\vec{\mathbf{r}}/dt$ but with length 1.



$\overrightarrow{\mathbf{T}}$ example

Example

Let $\vec{\mathbf{r}}(t) = (1 + 3\cos(t))\vec{\mathbf{i}} + (3\sin(t))\vec{\mathbf{j}} + t^2\vec{\mathbf{k}}$. *Find* $\overrightarrow{\mathbf{T}}$.

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Example

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We have

$$\vec{\mathbf{v}}(t) = \langle -3\sin(t), 3\cos(t), 2t \rangle.$$

Then

$$\|\vec{\mathbf{v}}\| = \sqrt{(-3\sin(t))^2 + (3\cos(t))^2 + (2t)^2} =$$

$$\sqrt{9\sin^2(t) + 9\cos^2(t) + 4t^2} = \sqrt{9 + 4t^2}.$$

Lastly, $\overrightarrow{\mathbf{T}}$ *is* $\vec{\mathbf{v}}$ *divided by* $\|\vec{\mathbf{v}}\|$.

$$\overrightarrow{\mathbf{T}} = \frac{\vec{\mathbf{v}}}{\|\vec{\mathbf{v}}\|} = \frac{\langle -3\sin(t), 3\cos(t), 2t \rangle}{\sqrt{9 + 4t^2}}$$

$$= \left\langle \frac{-3\sin(t)}{\sqrt{9 + 4t^2}}, \frac{3\cos(t)}{\sqrt{9 + 4t^2}}, \frac{2t}{\sqrt{9 + 4t^2}} \right\rangle.$$

Recall

Recall that if a vector-valued function $\vec{\mathbf{r}}$ always has a fixed length, then it has the property that $\vec{\mathbf{r}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = 0$.
As we showed,

$$\frac{d}{dt}\left[\vec{\mathbf{r}}(t) \cdot \vec{\mathbf{r}}(t)\right] = 0 \Rightarrow \frac{d\vec{\mathbf{r}}}{dt} \cdot \vec{\mathbf{r}}(t) + \vec{\mathbf{r}}(t) \cdot \frac{d\vec{\mathbf{r}}}{dt} = 0 \Rightarrow 2\vec{\mathbf{r}}(t)\frac{d\vec{\mathbf{r}}}{dt} = 0$$

which gives the desired result.

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Combining these facts, $\frac{d\vec{\mathbf{T}}}{dt}$ is orthogonal to the curve.

$\frac{d\vec{\mathbf{T}}}{dt}$ always points toward the 'inside' of the space curve, or in the direction that the curve is going.

Unit Normal vector

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The vector function

$$\overrightarrow{\mathbf{N}} = \frac{\frac{d\overrightarrow{\mathbf{T}}}{dt}}{\|\frac{d\overrightarrow{\mathbf{T}}}{dt}\|}$$

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To find $\overrightarrow{\mathbf{N}}$ given $\vec{\mathbf{r}}$, we calculate in order $\vec{\mathbf{v}}$, $\|\vec{\mathbf{v}}\|$, $\overrightarrow{\mathbf{T}}$, $d\overrightarrow{\mathbf{T}}/dt$, $\|d\overrightarrow{\mathbf{T}}/dt\|$, and lastly $\overrightarrow{\mathbf{N}}$.

$\overrightarrow{\mathbf{N}}$ example

Example

Let $\vec{\mathbf{r}}(t) = t\vec{\mathbf{i}} + \ln(\cos(t))\vec{\mathbf{j}}$, $-\pi/2 < t < \pi/2$. *Find* $\overrightarrow{\mathbf{N}}$.

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We have

$$\vec{\mathbf{v}}(t) = \vec{\mathbf{i}} + \frac{-\sin(t)}{\cos(t)}\vec{\mathbf{j}} \text{ and}$$

$$\|\vec{\mathbf{v}}(t)\| = \sqrt{1 + \frac{\sin^2(t)}{\cos^2(t)}} = \sqrt{1 + \tan^2(t)} = \sqrt{\sec^2(t)} = \sec(t)$$

$$(\text{because } -\pi/2 < t < \pi/2).$$

Thus we have

$$\vec{\mathbf{T}} = \frac{1}{\sec(t)}\vec{\mathbf{i}} - \frac{\tan(t)}{\sec(t)}\vec{\mathbf{j}} = \cos(t)\vec{\mathbf{i}} - \sin(t)\vec{\mathbf{j}}.$$

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Taking derivatives, we find $\frac{d\overrightarrow{\mathbf{T}}}{dt} = -\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}$ and $\left\|\frac{d\overrightarrow{\mathbf{T}}}{dt}\right\| = \sqrt{\sin^t(t) + \cos^2(t)} = 1$, so $\overrightarrow{\mathbf{N}} = -\sin(t)\vec{\mathbf{i}} - \cos(t)\vec{\mathbf{j}}$.

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Notice that since $\|\overrightarrow{\mathbf{B}}\| = \|\overrightarrow{\mathbf{T}}\|\|\overrightarrow{\mathbf{N}}\| \sin(\pi/2) = 1$, the vector $\overrightarrow{\mathbf{B}}$ is already a unit vector by definition, so we don't need to divide by its length.

$\overrightarrow{\mathbf{B}}$ example

Example

Take as above $\overrightarrow{\mathbf{T}} = \langle \cos(t), -\sin(t), 0 \rangle$ *and* $\overrightarrow{\mathbf{N}} = \langle -\sin(t), -\cos(t), 0 \rangle$*. Find* $\overrightarrow{\mathbf{B}}$*.*

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Take as above $\overrightarrow{\mathbf{T}} = \langle \cos(t), -\sin(t), 0 \rangle$ *and* $\overrightarrow{\mathbf{N}} = \langle -\sin(t), -\cos(t), 0 \rangle$. *Find* $\overrightarrow{\mathbf{B}}$.

We have

$$\overrightarrow{\mathbf{B}} = \overrightarrow{\mathbf{T}} \times \overrightarrow{\mathbf{N}} = \langle 0-0, 0-0, -\cos^2(t) - \sin^2(t) \rangle = \langle 0, 0, -1 \rangle.$$

In this case $\overrightarrow{\mathbf{B}}$ is a fixed vector, but in most cases $\overrightarrow{\mathbf{B}}$ will be a vector-valued function just like $\overrightarrow{\mathbf{T}}$ and $\overrightarrow{\mathbf{N}}$.

Example no.2

Find $\overrightarrow{\mathbf{T}}$, $\overrightarrow{\mathbf{N}}$, $\overrightarrow{\mathbf{B}}$ for $\vec{\mathbf{r}}(t) = \vec{\mathbf{r}}(t) = (e^t \cos(t))\vec{\mathbf{i}} + (e^t \sin(t))\vec{\mathbf{j}} + 2\vec{\mathbf{k}}$